Transfer Systems on the Partially Ordered Set $X_{n_+^+}$

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Abstract

In this paper, we investigate the patterns and properties of transfer systems in $X_{n_+^+}$, a poset with minimal and maximal added elements. Through the analysis of cases $n = 0$ through $n = 3$, we identify a consistent formula to calculate the number of transfer systems for any n and propose general rules governing their structure.

1 Introduction

The study of transfer systems provides insight into combinatorial properties and structural relationships within posets. By exploring the discrete poset X_n and its extension $X_{n_+^+}$, we uncover systematic patterns in the enumeration and classification of transfer systems.

1.1 Structure of the Paper

The paper is organized as follows. In Section 2, we define key concepts related to partially ordered sets and transfer systems. Section 3 introduces $X_{n_+^+}$ and discusses its structural properties. Section 4 presents the results of the enumeration of transfer systems for small values of n, leading to a general pattern. Finally, Section 5 provides conclusions and directions for future research.

2 Definitions

2.1 Partially Ordered Sets (Poset)

A relation $<$ on a set S is called **strict partial order** if:

1. $<$ is transitive: for all $a, b, c \in S$, if $a < b$ and $b < c$, then $a < c$.

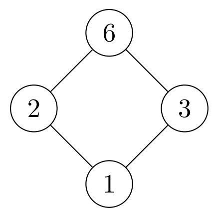
2. $<$ is antisymmetric: for all $a, b \in S$, if $a < b$, then $b \not< a$.

3. $<$ is irreflexive: for all $a \in S$, $a \not< a$.

If $<$ is a strict partial order, we can define $\leq$ on S by $a \leq b$ if and only if $a < b$ or $a = b$. Then $\leq$ is a partial order.

2.2 Meets (Lower Bound)

In a poset $(P, \leq)$, the **meet** of two elements a and b, denoted $a \wedge b$, is the greatest lower bound (GLB) of a and b.
Consider the poset $P = \{1, 2, 3, 6\}$ with the divisibility relation:



Consider the poset $P = \{1, 2, 3, 6\}$ with the divisibility relation. For $a = 2$ and $b = 3$, the meet is:

$$2 \wedge 3 = 1$$

2.3 Hasse Diagram

A Hasse diagram for a partially ordered set is a diagram with the following properties:

1. Each element of S is represented as a dot or its name in the diagram.
2. If $a < b$, the dot for a is lower than the dot for b.
3. If $a < b$, a line segment connects the dots for a and b.

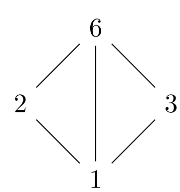


Figure 1: Hasse diagram of $P = \{1, 2, 3, 6\}$ with the divisibility relation.

2.4 Transfer Systems on Posets

[2] Let $P = (P, \leq)$ be a poset. A **transfer system** on P consists of a partial order R on P that refines $\leq$ and such that for a $x, y, z \in P$, if $x \; R \; y$, $z \leq y$, and

$x \wedge z$ exists, then $(x \wedge z)\ R\ z$.[1]

$$\text{If } x\ R\ y \text{ and } z \leq y, \text{ then } (x \wedge z)\ R\ z.$$

$$\begin{tikzcd}
x \wedge z \arrow[r, "R"] \arrow[d] \arrow[dr, phantom, "\uparrow R"] & x \arrow[d] \\
z \arrow[r, "R"'] & y
\end{tikzcd}$$

Figure 2: Transfer System Diagram with Relation, R, arrows

2.5 Minimal Fibrant Elements

Suppose P is a poset, R is a transfer system on P, P has a top element M, and there is a unique minimal fibrant element x. Then define:

$$B = [x, M] = \{y \in P \mid x \leq y \leq M\}.$$

If P has all meets and a top element M, any transfer system on P has a minimal fibrant element.

$$\begin{tikzcd}
0 & 1 \arrow[r] \arrow[rr, bend left, dash] & 2 & 3
\end{tikzcd}$$

$$\begin{tikzcd}
0 & 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 3
\end{tikzcd}$$

Figure 3: This is the partially ordered set [n] where n = 3. This is an example when 1 is the minimal fibrant.

2.6 Discrete Poset

If X is a set, we can always turn it into a poset by giving it the ordering known as the discrete ordering, which is the ordering $x \leq_{discrete} y$ if and only if x = y. We say that a poset P is discrete if the ordering on it is the discrete ordering.

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[1] The definition of transfer systems varies upon author. Our research revolves on partially order sets where we formulate the condition x ∧ z. In other research involving lattices, requires x ∧ z to be hold true.

2.7 Minimal and Maximal Elements

If P is a poset, we can also add new **maximum and minimum elements** in the following way. Let P^+ be the poset with underlying set $P \cup \{M\}$ with the relation $x \leq_{P^+} y$ if and only if $y = M$ or x and y are both in P and $x \leq_P y$. M will be the **maximum element** in P^+.
Similarly, we let P+ be the poset with underlying set $P \cup \{m\}$ with the relation $x \leq_{P^+} y$ if and only if $x = m$ or x and y are both in P and $x \leq_P y$. m will be the **minimum element** in P^+. We can also do both operations to the same poset and form P_+^+, which adds both a new maximum and a new minimum to P.

3 Background of $X_{n+}^{\ +}$

Let X_n be the discrete poset with n elements. Consider transfer systems on $X_{n+}^{\ +} = X_n \cup \{m, M\}$, where m is minimal and M is maximal.

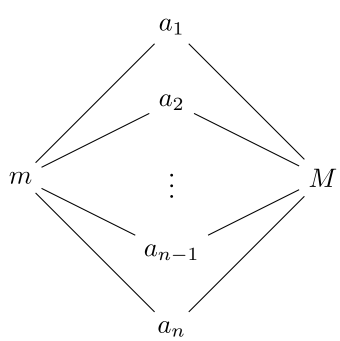

Figure 4: Hasse diagram of $X_{n+}^{\ +}$.

4 Results

We conclude with 2 transfer systems for case $n = 0$ which are shown below:

$$m \qquad\qquad M$$

$$m \xrightarrow{R} M$$

Figure 5: Hasse diagram of $X_{n+}^{\ +}$ when $n = 0$.

While we were searching for the number of transfer systems when $n = 1$ using the definition, we came up with rules for the relations in the poset. We also set each element in the set as a minimal fibrant element to obtain such transfer systems:

1. If $m \; R \; M$, then $m \; R \; a_n$ for all n.
2. If $a_n \; R \; m$, then $m \; R \; a_n$, for all $n \neq n$.
3. If $a_n \; R \; M$, then $m \; R \; a_i$ for all $i \neq n$.

We conclude with 5 transfer systems for case $n = 1$ which are shown below:

$$\begin{tikzcd} m & a_1 & M \end{tikzcd}$$

$$\begin{tikzcd} m \arrow[r, "R"] & a_1 & M \end{tikzcd}$$

$$\begin{tikzcd} m & a_1 \arrow[r, "R"] & M \end{tikzcd}$$

$$\begin{tikzcd} m \arrow[r, "R"] \arrow[rr, bend left, dash, "R"] & a_1 & M \end{tikzcd}$$

$$\begin{tikzcd} m \arrow[r, "R"] & a_1 \arrow[r, "R"] & M \end{tikzcd}$$

Figure 6: Transfer Systems of $X_{n_{+}^{+}}$ when $n = 1$.

We conclude with 10 transfer systems for case $n = 2$ which are shown below:

$$\begin{tikzcd} & a_1 & \\ m & & M \\ & a_2 & \end{tikzcd} \qquad \begin{tikzcd} & a_1 & \\ m \arrow[ur, "R"] & & M \\ & a_2 & \end{tikzcd}$$

$$\begin{tikzcd} & a_1 & \\ m \arrow[dr, "R"] & & M \\ & a_2 & \end{tikzcd} \qquad \begin{tikzcd} & a_1 & \\ m \arrow[ur, "R"] \arrow[dr, "R"] & & M \\ & a_2 & \end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 \arrow[dr, "R"] & \\
m \arrow[dr, "R"] & & M \\
 & a_2 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] & & M \\
 & a_2 \arrow[ur, "R"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[rr, "R"] \arrow[dr, "R"] & & M \\
 & a_2 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & a_1 \arrow[dr, "R"] & \\
m \arrow[ur, "R"] \arrow[rr, "R"] \arrow[dr, "R"] & & M \\
 & a_2 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[rr, "R"] \arrow[dr, "R"] & & M \\
 & a_2 \arrow[ur, "R"] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & a_1 \arrow[dr, "R"] & \\
m \arrow[ur, "R"] \arrow[rr, "R"] \arrow[dr, "R"] & & M \\
 & a_2 \arrow[ur, "R"] &
\end{tikzcd}$$

Figure 7: 10 transfer systems of $X_{n_+^+}$ for the case $n = 2$.

We conclude with 19 transfer systems for case $n = 3$ which are shown below:

$$\begin{tikzcd}
 & a_1 & \\
m & a_2 & M \\
 & a_3 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[r, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & a_1 & \\
m \arrow[dr, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[r, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[dr, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[r, "R"] \arrow[dr, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 \arrow[dr, "R"] & \\
m \arrow[r, "R"] \arrow[dr, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[dr, "R"] & a_2 \arrow[r, "R"] & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[r, "R"] & a_2 & M \\
 & a_3 \arrow[ur, "R"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] \arrow[rr, bend left, dash] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 \arrow[dr, "R"] & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] & a_2 & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] & a_2 \arrow[r, "R"] & M \\
 & a_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] & a_2 & M \\
 & a_3 \arrow[ur, "R"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_1 \arrow[dr, "R"] & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] & a_2 \arrow[r, "R"] & M \\
 & a_3 &
\end{tikzcd}$$

$$
\begin{tikzcd}
& a_1 \arrow[dr, "R"] & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] & a_2 & M \\
& a_3 \arrow[ur, "R"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& a_1 & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] & a_2 \arrow[r, "R"] & M \\
& a_3 \arrow[ur, "R"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& a_1 \arrow[dr, "R"] & \\
m \arrow[ur, "R"] \arrow[r, "R"] \arrow[dr, "R"] \arrow[rr, bend left] & a_2 \arrow[r, "R"] & M \\
& a_3 \arrow[ur, "R"] &
\end{tikzcd}
$$

Figure 8: 19 transfer systems of $X_{n_+^+}$ for the case $n = 3$.

We noticed a pattern in how to find the number systems for a certain number, n, in the discrete poset $X_{n_+^+}$. We noticed that when we set any element in the discrete set, a_n, as the minimal fibrant, we get one transfer system for each. For both m and M, we found that the transfer systems increase by a power of 2 as n increases. With this observation, we concluded with a formula that can provide us with the number of transfer systems as we increase the cardinality of the set:

$$2^{n+1} + n$$

From the examples above, we displayed the transfer system for $n = 0, 1, 2$ where we had 2, 5 and 10 transfer systems, which follow with the formula above.

5 Conclusion

In this study, we examined the structure of transfer systems on the poset $X_{n_+^+}$, consisting of a discrete set with added minimal and maximal elements. By systematically analyzing cases for small values of n, we observed a clear, consistent pattern in the enumeration of valid transfer systems. These patterns suggest a formulaic behavior in the number of possible systems as n increases, and we conjectured a general expression based on our findings.

Additionally, the structure of $X_{n_+^+}$ supports unique properties that simplify the verification of the transfer condition due to the discrete nature of the base set, X_n.The presence of a minimum and maximum element ensures the existence of meets when required, making $X_{n_+^+}$ a particularly rich candidate for exploring fibrant elements and their role in transfer systems.

Future directions for this research may involve a formal proof of the identified pattern, extending the approach to non-discrete posets, or exploring computational methods to enumerate transfer systems for larger n.

References

[1] Richard Johnsonbaugh, *Discrete Mathematics*, Eighth Edition, Pearson, DePaul University, Chicago, 2017.

[2] Evan E. Franchere, Kyle Ormsby, Angélica M. Osorno, Weihang Qin, and Riley Waugh, Self-duality of the lattice of transfer systems via weak factorization systems, arXiv e-prints (2021), arXiv:2102.04415. To appear in Homology, Homotopy, and Applications.

Appendix: C++ Program for Counting Transfer Systems

The following C++ code implements the formula $2^{n+1} + n$ to calculate the number of transfer systems for a discrete poset X_n^{++} of any given size n.

```
#include <iostream>
#include <cmath>
using namespace std;

int main(){
    int n = 0;
    char ch;

    do {
        cout << "Please enter number of elements in the set: ";
        cin >> n;
        cout << "The number of transfer systems for " << n
             << " elements is: " << (2*(pow(2,n))) + n << endl;
        cout << "Want to enter another number? (Y/N): ";
        cin >> ch;
    } while (toupper(ch) == 'Y');
}
```

This program allows users to explore the behavior of the formula and verify results for various values of n.